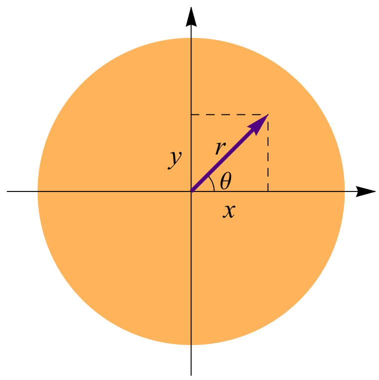
Lecture 8 - Angular Momentum

A Puzzle...

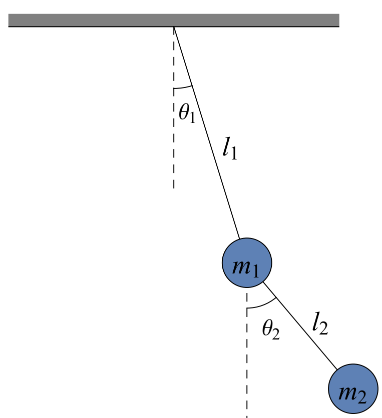
In computing the kinetic energy $\frac{1}{2} m v^2$, the velocity term v^2 is sometimes subtle. Suppose a particle moves with Cartesian coordinates x and y. Then the square of the particle's speed equals

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \tag{1}$$

1. What is the velocity in polar coordinates r and θ?



2. Sometimes, we can work in strange coordinate systems (we call these _generalized coordinates_). For example, what is the velocity of the bottom mass m_2 in the double pendulum shown below using θ_1 and θ_2 as the coordinates (in place of x and y)?



Solution

The secret to calculating v^2 is to always decompose any motion into its x and y components.

1. In polar coordinates, $x = r\,\text{Cos}[\theta]$ and $y = r\,\text{Sin}[\theta]$ so that

$$\frac{dx}{dt} = \frac{dr}{dt}\text{Cos}[\theta] - r\frac{d\theta}{dt}\text{Sin}[\theta] \tag{2}$$

$$\frac{dy}{dt} = \frac{dr}{dt}\text{Sin}[\theta] + r\frac{d\theta}{dt}\text{Cos}[\theta] \tag{3}$$

Therefore,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 \tag{4}$$

In this case, $\frac{dr}{dt}$ is the velocity in the radial direction, which is analogous to $\frac{dx}{dt}$ or $\frac{dy}{dt}$ but moves in the radial direction. $r \frac{d\theta}{dt}$ is the velocity in the polar direction; for example, a particle on a circle of radius r will travel in time t a distance $\int_0^t v_\theta \, dt = \int_0^t \left(r \frac{d\theta}{dt}\right) dt = \int_0^t r \, d\theta = r\,(\theta[t] - \theta[0])$ which is the familiar length of the arc between the point $(r, \theta[t])$ and $(r, \theta[0])$.

2. For the double pendulum, $x = l_1 \operatorname{Sin}[\theta_1] + l_2 \operatorname{Sin}[\theta_2]$ and $y = -l_1 \operatorname{Cos}[\theta_1] - l_2 \operatorname{Cos}[\theta_2]$ so that

$$\frac{dx}{dt} = -l_1 \frac{d\theta_1}{dt} \operatorname{Cos}[\theta_1] - l_2 \frac{d\theta_2}{dt} \operatorname{Cos}[\theta_2] \tag{5}$$

$$\frac{dy}{dt} = l_1 \frac{d\theta_1}{dt} \operatorname{Sin}[\theta_1] + l_2 \frac{d\theta_2}{dt} \operatorname{Sin}[\theta_2] \tag{6}$$

Therefore,

$$v^2 = l_1^2 \left(\frac{d\theta_1}{dt}\right)^2 + l_2^2 \left(\frac{d\theta_2}{dt}\right)^2 + 2\, l_1\, l_2 \frac{d\theta_1}{dt} \frac{d\theta_2}{dt} \operatorname{Cos}[\theta_1 - \theta_2] \tag{7}$$

This cross term, which is computed so naturally here, would be a pain to calculate by any other method. □

Angular Momentum

Basics

We define the <u>*angular momentum*</u> $\vec{L}$ about a point O as

$$\vec{L} \equiv \vec{r} \times \vec{p} \tag{8}$$

where $\vec{r}$ is the vector from O to the base of vector $\vec{p}$. (Note the similarity between angular momentum and torque, $\vec{\tau} = \vec{r} \times \vec{F}$, which is also defined about any point O where $\vec{r}$ is the vector from O to the base of vector $\vec{F}$.)

Angular momentum on a mass m with momentum $\vec{p} = m\,\vec{v}$ is related to its torque through

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \\ &= \vec{v} \times (m\,\vec{v}) + \vec{r} \times \vec{F} \\ &= \vec{r} \times \vec{F} \\ &= \vec{\tau} \end{aligned} \tag{9}$$

Therefore, when the net torque on a particle equals zero, its angular momentum is conserved.

If the angular momentum is changing over time, $\frac{d\vec{L}}{dt} \neq \vec{0}$, there are two possible causes: $\vec{L}$ either changes in magnitude or in direction (or both). If we define $\vec{L} = L\,\hat{L}$ then when $\vec{L}$ changes we either change L or $\hat{L}$ (or both).

In this course, we will deal primarily with the case where L changes but $\hat{L}$ remains static. An example of such a case is when you ride a carousel and your friends comes by and gives you a push, spinning you around even faster. These types of problems are particularly simple because we can essentially forget that $\vec{L}$ is a vector and only deal with its magnitude L.

To get a qualitative feel for the second case when $\hat{L}$ changes, let's consider the parallel situation with Newton's 2nd law $\vec{F} = \frac{d\vec{p}}{dt}$. Writing $\vec{p} = p\,\hat{p}$, the quantity $\frac{d\vec{p}}{dt}$ can change in two different ways: (1) the magnitude p can change while keeping the direction $\hat{p}$ constant (e.g., when an object accelerates in a line; in such cases, we can usually drop the vectors and simply write $F = m\,a$ along this line). Alternatively, (2) the direction $\hat{p}$ can change while keeping p constant (e.g., uniform circular motion; which gives rise to the centripetal acceleration $F = \frac{m\,v^2}{r}$).

The case where $\hat{p}$ changes is often significantly more complicated. Similarly, in cases where $\hat{L}$ changes, the dynamics can easily make your head *spin*!

Math Recap: Matrix Determinants

Recall that a useful pneumonic for the cross product is through the matrix determinant

$$\vec{r}\times\vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ p_x & p_y & p_z \end{vmatrix} \tag{10}$$

The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c \tag{11}$$

and that the determinant of a 3×3 matrix is given by expansion by minors,

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a\,e\,i - a\,f\,h - b\,d\,i + b\,f\,g + c\,d\,h - c\,e\,g \end{aligned} \tag{12}$$

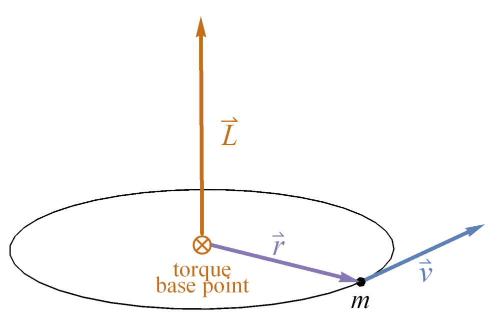
Therefore, the angular momentum is given by

$$\begin{aligned} \vec{L} &= \vec{r}\times\vec{p} \\ &= (r_y\,p_z - r_z\,p_y)\,\hat{x} - (r_x\,p_z - r_z\,p_x)\,\hat{y} + (r_x\,p_y - r_y\,p_x)\,\hat{z} \end{aligned} \tag{13}$$

Computing Angular Momentum

<u>Example</u>

A particle with mass m is undergoing uniform circular motion with velocity $\vec{v}$. What is its angular momentum as it moves around the circle?



<u>Solution</u>

In uniform circular motion, $\vec{r}$ is perpendicular to $\vec{v}$, and therefore

$$\begin{aligned} |\vec{L}| &= |\vec{r}\times\vec{p}| \\ &= m\,|\vec{r}\times\vec{v}| \\ &= m\,r\,v \end{aligned} \tag{14}$$

This statement holds for every point along the particle's circular motion, and therefore $|\vec{L}| = m\,r\,v$ is constant throughout the particle's circular motion. The direction of $\vec{L}$ is also constant, and by the property of the cross product it remains perpendicular to the plane defined by $\vec{r}$ and $\vec{v}$.

Another way to compute $\vec{L}$ is by writing out the Cartesian coordinates of $\vec{r}$ and $\vec{v}$ and then explicitly taking the cross product. Note that in time t the angle of the mass changes by $\frac{v}{r}t$. Align the $\hat{x}$- and $\hat{y}$-axes to point along $\vec{r}$ and $\vec{v}$ at $t=0$ so that

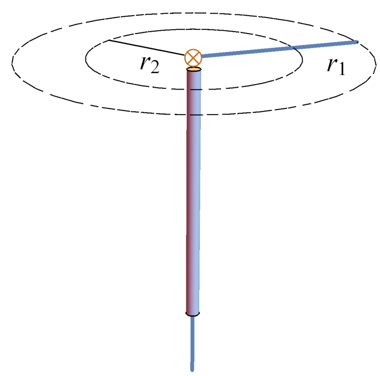
$$\vec{r} = \left\langle r\,\text{Cos}\left[\frac{v}{r}t\right], r\,\text{Sin}\left[\frac{v}{r}t\right], 0\right\rangle \tag{15}$$

$$\vec{v} = \left\langle -v\,\text{Sin}\left[\frac{v}{r}t\right], v\,\text{Cos}\left[\frac{v}{r}t\right], 0\right\rangle \tag{16}$$

We can now directly compute $\vec{L} = m\,\vec{r}\times\vec{v} = m\,r\,v\,\hat{z}$, which is indeed constant throughout the particle's motion. □

Example

A mass m is twirled in a circle of radius r_1 with a constant speed v_1. If the string is pulled so that the mass moves in a circle of radius r_2, what is the new velocity v_2 and angular velocity ω_2?



Solution

When the mass is circling with radius r_1, the inwards tension force T is what provides the centripetal acceleration. When this tension force is changed to bring the particles to r_2, its corresponding torque about the center of the particle's circular motion equals $\vec{\tau} = \vec{r}\times\vec{T} = \vec{0}$ because $\vec{r}$ and $\vec{T}$ are parallel. Therefore, $\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{0}$ and angular momentum is conserved.

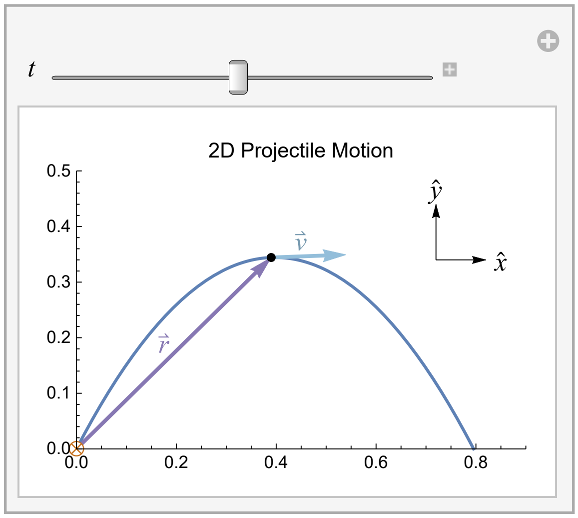
As we found above, the angular momentum of a mass m traveling in a circle of radius r_1 with velocity v_1 has magnitude $L = m\,r_1\,v_1$. The vector $\vec{L}$ will point in the direction $\vec{r}_1\times\vec{v}_1$, which must be the same direction as $\vec{r}_2\times\vec{v}_2$ since angular momentum is conserved. The magnitude of the angular momentum when the particle travels in a circle of radius r_2 with velocity v_2 equals $L = m\,r_2\,v_2$. Equating the angular momentum, we find $r_1\,v_1 = r_2\,v_2$ or

$$v_2 = \frac{r_1}{r_2}v_1 \tag{17}$$

Hence if $r_2 < r_1$ then $v_2 > v_1$, so that the velocity increases as the radius decreases. Conversely, if $r_2 > r_1$ and the radius increases, the velocity will decrease. The initial angular velocity equals $\omega_1 = \frac{v_1}{r_1}$ while the final angular velocity equals $\omega_2 = \frac{v_2}{r_2} = \frac{r_1}{r_2^2}v_1 = \left(\frac{r_1}{r_2}\right)^2\omega_1$ and therefore depends on the ratios of the radii squared. □

Example

A cannon shoots a projectile with velocity v_0 at angle θ. Is angular momentum relative to the cannon conserved? Why?



<u>Solution</u>

We write the velocity and position of the projectile as

$$\vec{v} = \langle v_0 \operatorname{Cos}[\theta],\ v_0 \operatorname{Sin}[\theta] - g\,t \rangle \tag{18}$$

$$\vec{r} = \left\langle v_0\, t \operatorname{Cos}[\theta],\ v_0\, t \operatorname{Sin}[\theta] - \tfrac{1}{2}\, g\, t^2 \right\rangle \tag{19}$$

At $t = 0$, $\vec{r} = \langle 0, 0 \rangle$ so that the angular momentum $\vec{L} = \vec{r} \times \vec{p} = \vec{0}$ (note that to carry out these cross products, we assume that all vectors have a 0 component in the z-direction).

At a later time t, we can visually see from the motion that $\vec{L} \neq \vec{0}$. Formally, we can calculate $\vec{L} = m\,\vec{r} \times \vec{v} = -\left(\frac{1}{2}\, m\, g\, v_0 \operatorname{Cos}[\theta]\, t^2\right) \hat{z}$ showing that the particle has angular momentum pointing into the page (we always use a right-handed coordinate system!)

Where does this angular momentum come from? It must come from the only other force in this problem: gravity. Indeed, the gravitational force $m\,\vec{g} = \langle 0, -m\,g \rangle$ creates a torque about $\vec{r}$ which equals $\vec{\tau} = \vec{r} \times \vec{F} = -m\,g\,t\,v_0 \operatorname{Cos}[\theta]\,\hat{z}$. Since $\vec{\tau} = \frac{d\vec{L}}{dt}$, we can integrate the torque to find $\vec{L} = \int \vec{\tau}\, dt$ up to a constant which will be determined by $\vec{L} = \vec{0}$ at $t = 0$. Integrating,

$$\begin{aligned} \vec{L} &= \int \vec{\tau}\, dt \\ &= \int -m\,g\,t\,v_0 \operatorname{Cos}[\theta]\,\hat{z}\, dt \\ &= -m\,g\,v_0 \operatorname{Cos}[\theta]\,\hat{z} \int t\, dt \\ &= -\left(\tfrac{1}{2}\, m\, g\, v_0 \operatorname{Cos}[\theta]\, t^2\right) \hat{z} + \vec{C} \end{aligned} \tag{20}$$

where $t = 0$ shows that $\vec{C} = 0$. Thus, we have confirmed that the gravitational force is responsible for the non-zero angular momentum of the particle. □

Swinging Your Arms

<u>Example</u>

You are standing on the edge of a step on some stairs, facing up the stairs. You feel yourself starting to fall backwards, so you start swinging your arms around in vertical circles, like a windmill. This is what people tend to do in such a situation, but does it actually help you not to fall, or does it simply make you look silly? Explain your reasoning.

<u>Solution</u>

Yes, swinging your arms, especially in a circular motion, helps you from rotating in the air. If you are starting to fall backwards, then your angular momentum (using your feet as the base point) points to your right. Circling your arms forward creates an angular momentum leftwards, which will help slow your rotation. The moral of the story: trust your instincts! □

Modern Research

This fantastic YouTube describes how toy models still exhibit very strange phenomena that we don't understand. The one calculation that is done requires moment of inertia, which we will learn about next time.

Advanced Section: Particle under Radial Force

Mathematica Initialization